Homework 6
MTH 869 Algebraic Topology

Joshua Ruiter

February 12, 2018

Proposition 0.1 (Exercise 1.3.13). *Consider the graph on the attached sheet (last page of this PDF), and denote it $\widetilde{X}$. Identify the left and right edges, so that it wraps around a cylinder. Then identify the top and bottom edges, wrapping it around a torus, using the double and triple arrows. Define a covering map $p : \widetilde{X} \to X = S^1 \vee S^1$ by sending each edge a to the a loop and each each b to the b loop. We claim that this covering space corresponds to the subgroup of $\pi_1(S^1 \vee S^1)$ generated by all cubes of elements.*

Proof. For the sake of argument, say the basepoint of $\widetilde{X}$ is the leftmost point. We can see that the loop a^3, going around an a-triangle, projects down to the identity, so $p_*(\pi_1(\widetilde{X}))$ has a^3 as a generator. Likewise, going around a b-triangle gives b^3 as a generator. Going around a hexagon gives $(ab)^3$ as a generator. Perhaps less obviously, it also has $(ab^{-1})^3$ as a generator. Starting at our basepoint (or any point), taking the a edge, then b edge backwards, etc. gets us back to our starting point because we wrap around the top. More generally, for any word in the free group on a, b, traversing that word three times on our graph gets us back to where we started, so $p_*(\pi_1(\widetilde{X}))$ is the desired subgroup. □

Lemma 0.2 (Exercise 1.3.14). *$\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to $\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$, where $\phi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ is the map $0 \mapsto \mathrm{Id}_\mathbb{Z}$ and $1 \mapsto (x \mapsto -x)$.*

Proof. We have the following presentation for $\mathbb{Z}_2 * \mathbb{Z}_2$:

$$\langle\, a, b \mid a^2, b^2 \,\rangle$$

In the semidirect product described, $\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$, let $a = (1, 1)$ and $b = (0, 1)$. Then

$$a + a = (1, 1) + (1, 1) = (1 + \phi(1)(1), 0) = (1 + -1, 0) = (0, 0)$$

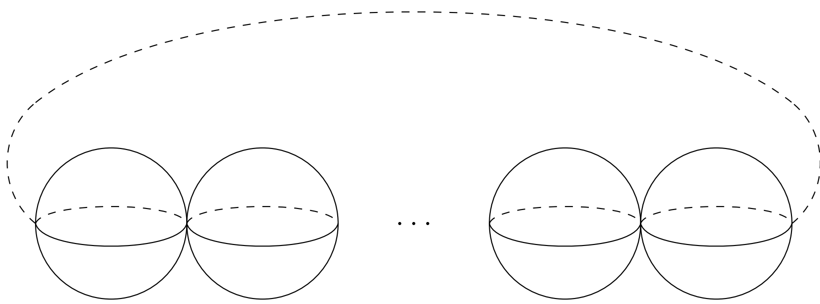
so $2a = 2b = 0$. (Here we're writing the operation in $\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$ additively, but we could also write $2a$ as a^2.) So $\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$ is generated by a, b, and we have the same relations as in $\mathbb{Z}_2 * \mathbb{Z}_2$, so they are isomorphic by $a \mapsto a$ and $b \mapsto b$. □

Lemma 0.3 (Exercise 1.3.14). *Define $\phi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ by $0 \to \mathrm{Id}$ and $1 \mapsto (x \mapsto -x)$. All subgroups of $\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$ are of the form $0, \mathbb{Z}_2, n\mathbb{Z}$, or $n\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$ where $n \in \mathbb{N}$. There is a subgroup of each type for every $n \in \mathbb{N}$.*

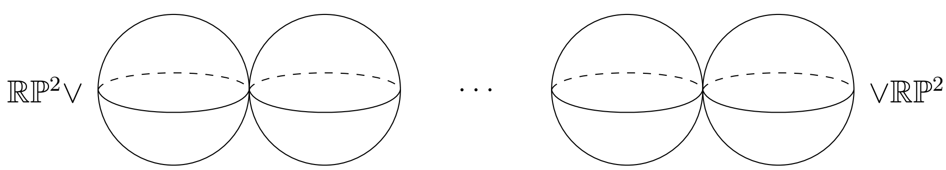
Proof. Let $G = \mathbb{Z} \rtimes_\phi \mathbb{Z}_2$ and let $H \subset G$ be a subgroup. Assume H is not trivial and $H \neq \langle (0,1) \rangle$. Then there exists $(n,0) \in H$ with n positive, and we choose n so that $|n|$ is minimized (and still positive). If $(0,1) \notin H$, then H is the cyclic subgroup generated by $(n,0)$, which is of the form $n\mathbb{Z}$. If $(0,1) \in H$, then H is generated by $(n,0)$ and $(0,1)$, so H has the form $n\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$.

Now we show there is a subgroup of each type for every $n \in \mathbb{N}$. The subgroup generated by $(0,1)$ is $\mathbb{Z}_2$. The subgroup generated by $(n,0)$ is $n\mathbb{Z}$. The subgroup generated by $(n,0)$ and $(0,1)$ is $n\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$. □

Proposition 0.4 (Exercise 1.3.14)**.** *The path connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ are itself, its universal cover, and covering spaces of the following two forms. First, chains of $2n$ 2-spheres forming a looped "necklace" of $2n$ beads.*



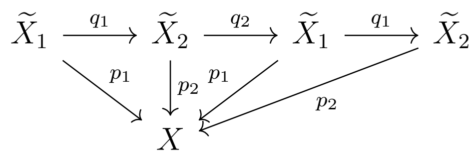
The second type is a chain of n 2-spheres with a copy of $\mathbb{RP}^2$ wedged at each end (with $n = 0$ or $n \in \mathbb{N}$).



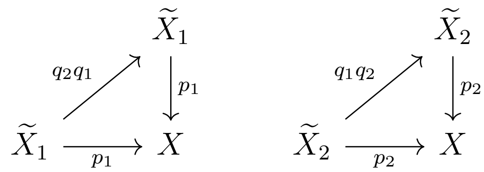
Proof. As shown above, all subgroups of $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2)$ are of the form $0, \mathbb{Z}_2, n\mathbb{Z}$, or $n\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$ for $n \in \mathbb{N}$. The trivial subgroup corresponds to the universal covering, depicted in Example 1.48 (Hatcher pg. 78). The necklace with $2n$ copies of S^2 covers $\mathbb{RP}^n \vee \mathbb{RP}^n$ similarly to the universal cover, so the image subgroup is generated by $(ab)^n$, which is the subgroup $n\mathbb{Z}$ of $\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$. The covering of n copies of S^2 with a copy of $\mathbb{RP}^2$ at each end induces the subgroup generated by a and a word of the form $b, bab, babab, \ldots$ which corresponds to being generated by $(0,1)$ and $(n,0)$, so the subgroup is $n\mathbb{Z} \rtimes_\phi \mathbb{Z}_2$. Note that this holds for $n = 0$ as well. □

Lemma 0.5 (for Exercise 1.3.18)**.** *Let $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ be covering maps so that there is a covering map $q_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ that is a lift of p_1 and a covering map $q_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$ that is a lift of p_2. Then $\widetilde{X}_1, \widetilde{X}_2$ are isomorphic as (based) covers.*

Proof. We have the following commutative diagram,



from which we can extract the following commutative triangles:



This says that q_2q_1 and q_1q_2 are respective lifts of p_1 and p_2. But another lift of p_1 is $\mathrm{Id}_{\widetilde{X}_1}$, and another lift of p_2 is $\mathrm{Id}_{\widetilde{X}_2}$. Since $q_2q_1(\widetilde{x}_1) = \widetilde{x}_1$ and $q_1q_2(\widetilde{x}_2) = \widetilde{x}_2$, by uniqueness of lifting we get $q_2q_1 = \mathrm{Id}_{\widetilde{X}_1}$ and $q_1q_2 = \mathrm{Id}_{\widetilde{X}_2}$. Thus q_2q_1 is an isomorphism of (based) covers. □

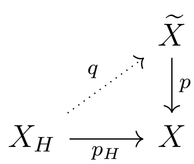
Proposition 0.6 (Exercise 1.3.18, part one)**.** *Let X be path connected, locally path connected, and semilocally simply connected. Then X has an abelian covering space that is a cover of every other abelian covering space of X. This universal abelian covering space is unique up to isomorphism.*

Proof. First we construct the universal abelian cover. Let $H \subset \pi_1(X)$ be the commutator subgroup. By Proposition 1.36, there is a covering space $p_H : X_H \to X$ so that $(p_H)_*(\pi_1(X_H)) = H$. By Proposition 1.39, since H is a normal subgroup, so $p_H : X_H \to X$ is a normal covering, and $G(X_H) \cong \pi_(X)/H$ is abelian, so X_H is a normal covering space.

Now let $p : \widetilde{X} \to X$ be any abelian covering space of X. Let $K = p_*(\pi_1(\widetilde{X}))$. Then K is a normal subgroup of $\pi_1(X)$, and $G(\widetilde{X}) \cong \pi_1(X)/K$ is abelian. This implies that $H \subset K$, since the commutator subgroup is the smallest subgroup so that $\pi_1(X)/H$ is abelian. Then

$$(p_H)_*(\pi_1(\widetilde{X})) = H \subset K = p_*(\pi_1(X))$$

by Proposition 1.33, there is a lift q so that the following diagram commutes.

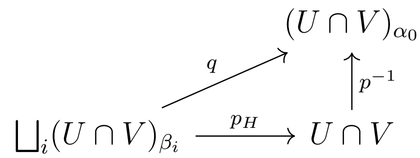


We claim that q is a covering map. Let $\widetilde{x} \in \widetilde{X}$. We need to find a neighborhood of $\widetilde{x}$ that is evenly covered (with respect to q). Let $x = p(\widetilde{x})$. Since X_H and $\widetilde{X}$ are covering spaces, there exist evenly covered neighborhoods U, V of x with $x \in U \cap V$ where U is evnely covered with respect to p and V is evenly covered with respect to X_H. Then

$$p^{-1}(U \cap V) = \bigsqcup_\alpha (U \cap V)_\alpha \subset \widetilde{X}$$

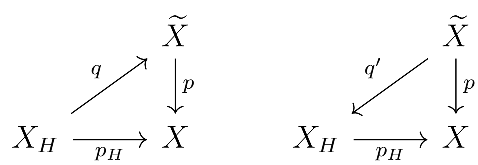
$$p_H^{-1}(U \cap V) = \bigsqcup_\beta (U \cap V)_\beta \subset X_H$$

Choose the unique α_0 so that $\widetilde{x} \in (U \cap V)_{\alpha_0}$. The set $(U \cap V)_{\alpha_0}$ is our candidate for an evenly covered neighborhood of $\widetilde{x}$. Note that $p|_{(U \cap V)_{\alpha_0}} : (U \cap V)_{\alpha_0} \to U \cap V$ is a homeomorphism; denote its inverse by p^{-1}. Since q is a lift of p, q maps $p_H^{-1}(U \cap V)$ to $p^{-1}(U \cap V)$, so some nonempty subset of $p_H^{-1}(U \cap V)$ gets mapped to $(U \cap V)_{\alpha_0}$ (by q).



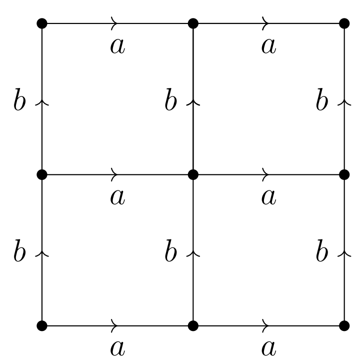
On $\bigsqcup_i (U \cap V)_{\beta_i}$, we have that $q = p^{-1} p_H$, so $q^{-1}((U \cap V)_{\alpha_0}) = \bigsqcup_i (U \cap V)_{\beta_i}$, and $q = p^{-1} p_H$ maps each $(U \cap V)_{\beta_i}$ homeomorphically to $(U \cap V)_{\alpha_0}$ since p^{-1} is a homeomorphism and p_H maps each $(U \cap V)_{\beta_i}$ homeomorphically to $U \cap V$. Thus q is a covering map. This concludes the proof that X_H is a cover of every other abelian covering of X.

Now for uniqueness. Suppose that $p : \widetilde{X} \to X$ is another abelian covering space that covers every abelian covering space of X. Then there are covers $q : \widetilde{X} \to X_H$ and $q' : X_H \to \widetilde{X}$ that make the following diagrams commute.



Then by the previous lemma, $\widetilde{X}$ and X_H are isomorphic as covers. Hence X_H is unique up to isomorphism. $\square$

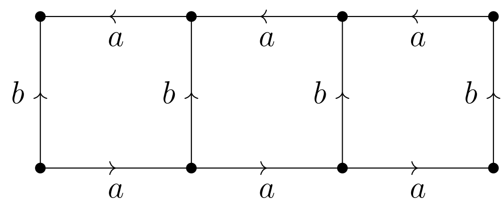
Proposition 0.7 (Exercise 1.3.18, part two). *The universal abelian cover for $S_1 \vee S_1$ is an infinite two dimensional lattice, a piece of which is depicted below:*



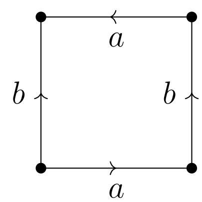
The universal abelian cover for $S_1 \vee S_1 \vee S_1$ is a three dimensional infinite lattice generated by a, b, c.

Proof. The universal abelian covering of $S^1 \vee S^1$ corresponds to the commutator subgroup of $\mathbb{Z} * \mathbb{Z}$. After taking the quotient by the commutator subgroup, we get a cover with fundamental group $\mathbb{Z} \times \mathbb{Z}$. In the case of $S^1 \vee S^1 \vee S^1$, we take the quotient of $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ by its commutator subgroup, so the universal abelian cover has fundamental group $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. $\square$

Proposition 0.8 (Exercise 1.3.20, part one)**.** *The following represents a non-normal covering of the Klein bottle by itself. We "stretch" the fundamental polygon of the Klein bottle horizontally and insert two more vertical lines.*



Then we map this to the original "square" representation of the Klein bottle, drawn below, by sending each line labelled a to the line a and each line b to the line b.



Proof. Let K denote the Klein bottle, and denote the above map by $p : K \to K$. It is a covering map by construction. We need to show that it is not a normal covering. We know that the fundamental group $\pi_1(K)$ has the presentation

$$\pi_1(K) \cong \langle\, a, b \mid abab^{-1} = e \,\rangle$$

since K is built by attaching a 2-cell to the above depicted fundamental polygon. The induced map $p_* : \pi_1(K) \to \pi_1(K)$ sends b to itself, but maps a to a^3, so the subgroup corresponding to this cover is

$$p_*(\pi_1(K)) = \langle\, a^3, b \mid abab^{-1} = e \,\rangle$$

We claim that this subgroup is not normal in $\pi_1(K)$. If it were normal, then we would have $aba^{-1} \in p_*(\pi_1(K))$, but

$$aba^{-1} = a(aba)a^{-1} = a^2baa^{-1} = a^2b$$

But a^2b is not in $p_*(\pi_1(K))$, so this is not a normal subgroup. Thus by Proposition 1.30 (Hatcher pg. 71) this is a non-normal covering. ☐

Definition 0.1. Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a universal covering space. Then the **action via lifting** of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ is the group action

$$\pi_1(X, x_0) \times p^{-1}(x_0) \to p^{-1}(x_0) \qquad ([\gamma], \widetilde{x}) \mapsto [\gamma]_L \widetilde{x} = \widetilde{\gamma}(1)$$

where $\widetilde{\gamma}$ is the unique lift of γ satisfying $\widetilde{\gamma}(0) = \widetilde{x}$.

Definition 0.2. Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a universal covering space, with $G(\widetilde{X})$ the group of deck transformations. Given $[\gamma] \in \pi_1(X, x_0)$, there is a unique lift $\widetilde{\eta}$ with $\widetilde{\eta}(0) = \widetilde{x}_0$. There is an isomorphism

$$\pi_1(X, x_0) \to G(\widetilde{X}) \qquad [\gamma] \mapsto \phi_\gamma$$

where ϕ_γ is the unique deck transformation such that $\phi_\gamma(\widetilde{x}_0) = \widetilde{\eta}(1)$. The **action via deck transformations** of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ is

$$\pi_1(X, x_0) \times p^{-1}(x_0) \to p^{-1}(x_0) \qquad ([\gamma], \widetilde{x}) \mapsto [\gamma]_D \widetilde{x} = \phi_\gamma(\widetilde{x})$$

Proposition 0.9 (Exercise 1.3.27, part one). *Let $X = S^1 \times S^1$ and let $p : \mathbb{R}^2 \to S^1 \times S^1$ be the universal cover. For this covering space, the action via lifting and the action via deck transformations are the same.*

Proof. We have $\pi_1(X)$ abelian, so this is a special case of the more general statement below. □

Proposition 0.10 (Exercise 1.3.27, part two). *Let $X = S^1 \vee S^1$ with basepoint x_0, and let $p : \widetilde{X} \to X$ be the universal cover. The action via lifting is not the same as the action via deck transformations.*

Proof. Let $F_{a,b}$ denote the free group generated by a, b, so $\pi_1(S^1 \vee S^1) = F_{a,b}$. Then $\widetilde{X}$ is the Cayley graph of $F_{a,b}$. We denote the vertices of $\widetilde{X}$ by elements of the free group, in the following way, by denoting the vertex reached by traversing a word $w \in F[a, b]$ by $\widetilde{x}_w$, so we have

$$p^{-1}(x_0) = \{\widetilde{x}_w : w \in F_{a,b}\}$$

In particular, we denote $\widetilde{x}_0$ by $\widetilde{x}_e$, where e is the identity. Let $a = [\gamma] \in \pi_1(X, x_0)$. Using the action via lifting, we get

$$[\gamma]_L \widetilde{x}_w = \widetilde{\gamma}(1)$$

where $\widetilde{\gamma}$ is the unique lift of a starting at $\widetilde{x}_w$. Looking locally at $\widetilde{X}$ at $\widetilde{x}_w$, the lift $\widetilde{\gamma}$ must be just the path going along the path a from $\widetilde{x}_w$, so

$$[\gamma]_L \widetilde{x}_w = \widetilde{x}_{wa}$$

On the other hand, using the action from deck transformations,

$$[\gamma]_D \widetilde{x}_w = \phi_{\gamma_a}(\widetilde{x}_w)$$

where ϕ_γ is the unique deck transformation mapping $\widetilde{x}_e$ to $\widetilde{\eta}(1)$, where $\widetilde{\eta}$ is the unique lift of γ so that $\widetilde{\eta}(0) = \widetilde{x}_e$. The unique lift of γ starting at $\widetilde{x}_e$ is the path from $\widetilde{x}_e$ to $\widetilde{x}_a$, so ϕ_γ is the unique deck transformation mapping $\widetilde{x}_e$ to $\widetilde{x}_a$. Since this deck transformation maps $\widetilde{x}_e$ to $\widetilde{x}_a$, it must also map the neighboring vertices of $\widetilde{x}_e$ to the neighbors of $\widetilde{x}_a$, preserving paths, so

$$[\gamma]_D \widetilde{x}_b = \phi_\gamma(\widetilde{x}_b) = \widetilde{x}_{ab}$$

In general, we see that

$$[\gamma]_D \widetilde{x}_w = \widetilde{x}_{aw}$$

So the action via lifting $[\gamma] = a$ is acting on $F_{a,b}$ by multiplying by a on the right, while the action via deck transformations is acting on $F_{a,b}$ by multiplying by a on the left. These two actions are not the same, since $ab \neq ba$. More concretely, we have shown that

$$[\gamma]_L \widetilde{x}_b = \widetilde{x}_{ba} \neq \widetilde{x}_{ab} = [\gamma]_D \widetilde{x}_b$$

so the actions are not the same. □

Proposition 0.11 (Exercise 1.3.27). *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a universal cover. There are two actions of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$, given by lifting loops at x_0, and the action of restricting deck transformations to the fiber. When $\pi_1(X, x_0)$ is abelian, these actions always agree.*

The following is an incomplete proof. Let $\widetilde{x} \in p^{-1}(x_0)$, and let $[\gamma] \in \pi_1(X, x_0)$ and, and choose a representative loop $\gamma : I \to X$ based at x_0. There are unique lifts $\widetilde{\gamma}, \widetilde{\eta}$ so that $\widetilde{\gamma}(0) = \widetilde{x}$ and $\widetilde{\eta}(0) = \widetilde{x}_0$. Let ϕ_γ be the unique deck transformation satisfying $\phi_\gamma(\widetilde{x}_0) = \widetilde{\eta}(1)$. Let $\widetilde{\alpha}$ be the unique lift of γ such that $\widetilde{\alpha}(0) = \phi_\gamma(\widetilde{x}_0)$. By definition,

$$\begin{aligned} [\gamma]_L \widetilde{x} &= \widetilde{\gamma}(1) & [\gamma]_D \widetilde{x} &= \phi_\gamma(\widetilde{x}) \\ [\gamma]_D [\gamma]_L \widetilde{x} &= \phi_\gamma(\widetilde{\gamma}(1)) & [\gamma]_L [\gamma]_D \widetilde{x} &= \widetilde{\alpha}(1) \end{aligned}$$

Let $\widetilde{\beta}$ be the unique (up to homotopy) path from $[\gamma]_L \widetilde{x}$ to $[\gamma]_L [\gamma]_D \widetilde{x}$. Let $\widetilde{\psi}$ be the unique (up to homotopy) path from $\widetilde{x}$ to $\phi_\gamma(\widetilde{x})$. Since the endpoints are in $p^{-1}(x_0)$, both $\widetilde{\beta}, \widetilde{\psi}$ project down to loops $[\beta], [\psi] \in \pi_1(X, x_0)$. In particular, we have the loop

$$\left[\widetilde{\gamma} \cdot \widetilde{\beta} \cdot \overline{\widetilde{\alpha}} \cdot \overline{\widetilde{\psi}}\right] \in \pi_1(\widetilde{X}, \widetilde{x}_0)$$

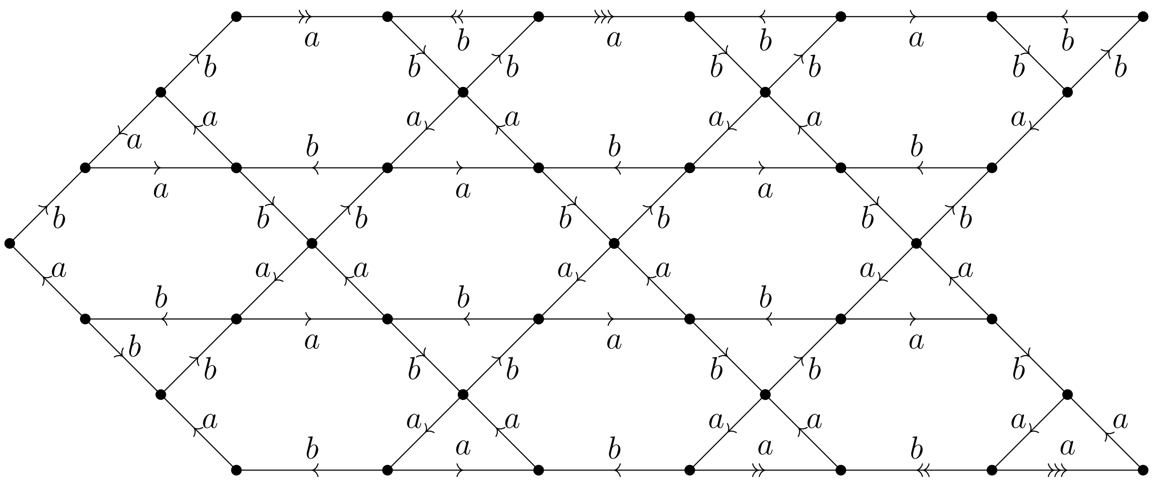
which projects down to

$$[\gamma][\beta][\gamma]^{-1}[\psi]^{-1} \in \pi_1(X, x_0)$$

since both $\widetilde{\alpha}$ and $\widetilde{\gamma}$ are lifts of γ. Since $\widetilde{X}$ is a universal cover, all loops are trivial, so this image loop is trivial. Since $\pi_1(X, x_0)$ is abelian, this implies

$$1 = [\gamma][\beta][\gamma]^{-1}[\psi]^{-1} \implies [\gamma][\gamma]^{-1}[\beta][\psi]^{-1} \implies [\beta] = [\psi]$$

I think this should give us what we want, but I'm not sure how to finish it.


a
b
a
b
a
b
b
b
b
b
b
b
a
a
a
a
a
a
a
b
b
b
a
a
a
b
b
b
b
b
b
a
a
a
a
a
a
a
b
b
b
a
a
a
b
b
b
b
b
b
a
a
a
a
a
a
a
a
b
b
b
a
a